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In this paper the author deals with the problem of constructing the field of fractions of an integral domain and a generalization of one of the methods of construction used to construct a ring of left quotients for an arbitrary ring. In this generalization the author relies heavily upon the concept of a faithful complete filter and defines partial endomorphisms from the filter elements into the ring. After partitioning these partial endomorphisms into equivalence classes and after defining operations on the equivalence classes the author then shows that the resultant structure is a ring of left quotients.

In addition to showing that a faithful complete filter assures the existence of a ring of left quotients the author shows that these rings of left quotients can be embedded in the Utumi ring of left quotients.

The paper concludes with theorems showing necessary and sufficient conditions for the classical ring of left quotients of a ring R to exist and a theorem establishing the uniqueness up to isomorphism of the classical ring of left quotients of a ring R.

A GENERALIZATION OF THE FIELD OF FRACTIONS

OF AN INTEGRAL DOMAIN

by

Robert Eugene Michaud

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E. E. Posey

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Adviser: E. E. Posey

Oral Examination Committee Members: Robert L. Bernhardt

Hughes B. Hoyle, III

Kenneth A. Byrd

Robert P. Miller

August 7, 1970

Date of Examination

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INTRODUCTION AND PRELIMINARY REMARKS

There is a well-known method for constructing the rational numbers Q^* from the integers Z; one defines $Q = \{(a,b)$: a,b are in Z and $b \neq 0\}$ and a relation $\sim$ between members of Q as follows: $(a,b) \sim (c,d)$ if and only if $ad = bc$. After showing that $\sim$ is an equivalence relation it follows that $\sim$ induces a partition of Q into equivalence classes; we denote the equivalence class containing (a,b) by $[a,b]$. The next step in the construction is to define a set $Q^* = \{[a,b]:(a,b)$ belongs to $Q\}$ and two operations, addition and multiplication, on members of Q^*. It is then an easy but tedious task to verify that $(Q^*,+,\cdot)$ is a field.

In chapter one, we shall use this method to construct the field of fractions of any integral domain. However, this construction cannot be generalized to arbitrary rings and so we use a second method to arrive at a field which will be shown to be isomorphic to the first.

In chapter two we will generalize the second method to construct a ring of left quotients for any ring and thus exhibit an advantage the second method has over the first, since the first method will not generalize to an arbitrary ring. In fact we shall show how to construct a set of left quotient rings for a ring R, and from this set we shall single out one of the most commonly studied rings, the Utumi or <u>maximal</u> ring of left quotients [1].

In chapter three we define a "classical ring of left quotients." This concept is the natural generalization of the first construction in chapter one to rings which are not integral domains and, as might be expected upon imposing a special condition, we will have to restrict the type of ring that we will discuss. The paper is concluded with theorems establishing the uniqueness of the classical ring of left quotients of a ring, as well as necessary and sufficient conditions for this quotient ring to exist.

In keeping with recent trends in the literature we wish to introduce the following definitions whose presence here permit better continuity of the text later.

We define a <u>left</u> <u>R-module</u> as an abelian group $(M,+)$ together with a function $\mu : R \times M \to M$, where we let $\mu(a,x) = a \mu x$, such that:

(a) $(a+b) \mu x = a \mu x + b \mu x$ for all a,b in R and for all x in M.

(b) $(a \cdot b) \mu x = a \mu (b \mu x)$ for all a,b in R and for all x in M.

(c) $a \mu (x+y) = a \mu x + a \mu y$ for all a in R and for all x,y in M.

(d) $1 \mu x = 1$ for all x in M.

We note that if a ring R has no unity we simply delete property (d) above. In either case we shall denote the left R-module M by ${}_RM$. If N is a subgroup of M then N is itself a candidate for a left R-module. We shall call a subgroup N of the group M a submodule of the left R-module ${}_RM$ whenever $a \mu x$ belongs to N for all a in R and x in N.

Whenever it is convenient we shall write the simpler ax instead of $a \mu x$. Written in this way it appears as though a module ${}_RM$ has two operations defined on it; addition in the group and the "product" of elements in the ring with elements in the group. It therefore seems natural to require that a module homomorphism preserves both of these "operations".

If ${}_RM$ and ${}_RL$ are left R-modules a function $f : M \to L$ is called a <u>left</u> <u>R-homomorphism</u> from M into L if in addition to being a group homomorphism it satisfies the following: $f(a\mu_1 x) = a\mu_2 f(x)$ for all a in R and x in M. We shall denote the set of all left R-homomorphisms from the left R-module ${}_RA$ into the left R-module ${}_RB$ by $\mathrm{Hom}_R(A,B)$.

Finally we define the following: If ${}_RN$ is a submodule of ${}_RM$ we say that ${}_RN$ is <u>essential</u> in ${}_RM$ if every nonzero submodule ${}_RL$ of ${}_RM$ intersects ${}_RN$ non-trivially and we write ${}_RN \trianglelefteq {}_RM$.

CHAPTER I

FIELD OF FRACTIONS OF AN INTEGRAL DOMAIN

In the discussion that follows we will construct a field which we will call the <u>field of fractions</u> of the integral domain R. No formal definition of this term will be given at this time and for the present we will assume that a field of fractions is the result of the construction that follows. We recall that a commutative ring R with more than one element and having a unity 1 is called an <u>integral domain</u> if for all a and b in R, if $ab = 0$, then $a = 0$ or $b = 0$. If R is a ring and d is a nonzero element of R we shall call d a <u>nonzero divisor</u> of R whenever $ad = 0$ or $da = 0$ implies $a = 0$.

Throughout the reaminder of this paper we shall denote the set of all nonzero divisors of a ring R by U(R). Since all the rings that we discuss are rings with unity, it is clear that $U(R) \neq \phi$. We shall denote the set of all ordered pairs (a,b) with a in R and b in U(R) by Q(R). With these preliminary notes we are now in a position to begin the construction of the field of fractions of an integral domain. For the remainder of this chapter we will deal exclusively with rings which are integral domains; thus R denotes an integral domain below.

<u>DEFINITION 1.1</u>: Let $\sim$ be a relation defined on Q(R) by $(a,b) \sim (c,d)$ if and only if $ad = bc$.

PROPOSITION 1.2: $\sim$ is an equivalence relation.

Proof: One easily checks that the reflexive and symmetric laws hold and so we prove only that $\sim$ is transitive. If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $ad = bc$ and $cf = de$. Thus $adf = bcf$ and $bcf = bde$, so $adf = bde$ and $af = be$ since d is not a divisor of zero. Hence $(a,b) \sim (e,f)$.

As a result of proposition 1.2 it is immediate that $\sim$ induces a partition of $Q(R)$ into equivalence classes; we denote by $[a,b]$ the equivalence class containing (a,b). That is, $[a,b] = \{(x,y) \text{ in } Q(R)\colon (a,b) \sim (x,y)\}$. We shall denote the set of all the equivalence classes by Q^*.

We wish to define two operations on Q^* as follows: (a) $[a,b] + [c,d] = [ad+bc,bd]$, (b) $[a,b][c,d] = [ac,bd]$.

It is clear that since bd belongs to $U(R)$ that both $(ad+bc,bd)$ and (ac,bd) are in $Q(R)$ and hence $[ad+bc,bd]$ and $[ac,bd]$ are in Q^*.

PROPOSITION 1.3: Addition and multiplication are well-defined.

Proof: Let $[a,b]$, $[c,d]$, $[e,f]$, and $[g,h]$ belong to Q^* with $[a,b] = [c,d]$ and $[e,f] = [g,h]$. Then $ad = bc$ and $eh = fg$. It is easy to check that if we multiply both sides of the first equality by fh and both sides of the second equality by bd and add that we obtain $(af+be)dh = (ch+gd)bf$. Thus $(af+be,bf) \sim (ch+gd,dh)$ and so $[a,b] + [e,f] = [c,d] + [g,h]$. Also $adeh = bcfg$ and so it is clear that $(ae,bf) \sim (cg,dh)$ and $[a,b] \cdot [e,f] = [c,d] \cdot [g,h]$.

PROPOSITION 1.4: $(Q^*,+,\cdot)$ is a field.

Proof: Let $[a,b]$, $[c,d]$, and $[e,f]$ belong to Q^*. Clearly addition and multiplication are commutative; also $[0,1]$ and $[1,1]$ are the additive and multiplicative identities, respectively, in Q^*. If $[a,b]$ is in Q^*, then $[-a,b] + [a,b] = [0,b] = [0,1]$. The associative laws are tedious and straightforward to show. Clearly if $[a,b]$ is in Q^* and $[a,b] \neq [0,1]$ then $a \neq 0$ and hence a is in $U(R)$ and $[b,a]$ is in Q^*. Further $[a,b] \cdot [b,a] = [1,1]$. We have shown that $(Q^*,+)$ and $(Q^*-\{[0,1]\},\cdot)$ are abelian groups. We are through as soon as we establish the distributive laws. $([a,b] + [c,d]) \cdot [e,f] = [ad+bc,bd] \cdot [e,f] = [ade+bce,bdf]$. We multiply this last expression by $[f,f] = [1,1]$ to get $[adef+bcef,bdf] = [ae,bf] + [ce,df] = [a,b] \cdot [e,f] + [c,d] \cdot [e,f]$.

Having shown that $(Q^*,+,\cdot)$ is a field, we will next show that Q^* contains a subring isomorphic to R. In fact we claim that $S = \{[a,1]$ in $Q^*\}$ is that subring. That S is a subring of Q^* is not hard to check directly but we will obtain this result while establishing a ring isomorphism between R and S. We define $f : R \to S$ by $f(a) = [a,1]$ for all a in R. Clearly $a = b$ implies that $(a,1) \sim (b,1)$ which in turn implies that $[a,1] = [b,1]$ and so f is well-defined. It is not hard to check that each of the above implications is reversible and we also have that f is one-to-one. Clearly f is onto and we check that f

is a ring homomorphism. If a and b are in R, then $f(a+b) = [a+b,1] = [a,1] + [b,1] = f(a) + f(b)$. Also $f(ab) = [ab,1] = [a,1] \cdot [b,1] = f(a) \cdot f(b)$.

The reader should note that if d is in $U(R)$ then $f(d)^{-1}$ is in Q^*, since $f(d)^{-1} = [1,d]$, in Q^*. Hence we know that for all a in R and d in $U(R)$, the element $[a,d]$ in Q^* looks like a fraction in the following sense: if the elements in R are identified with their images under f in S, then $[a,d] = [a,1] \cdot [1,d] = ad^{-1}$ in Q^*. It is this characterization that we shall generalize to a field of fractions of a ring.

We now procede to construct a field of fractions of R in a second way, concluding with a theorem which establishes an isomorphism between the results of the two methods.

<u>DEFINITION 1.5</u>: If S is a ring and I a left ideal of S we call an S-homomorphism $f : I \to S$ a <u>partial</u> <u>S-endomorphism</u>. We denote by $\mathrm{Hom}_S(A,B)$ the set of S-homomorphisms whose domain is the left S-module A and whose range is the left S-module B.

If R is an integral domain and b is in R, then a partial R-endomorphism f whose domain is Rb has as its image the principal ideal $Rf(b)$ and $f(rb) = rf(b)$. Hence we define f_{ab} in $\mathrm{Hom}_R(Rb,Ra)$ by $f_{ab}(rb)$ is equal to ra for each rb in Rb, so that $a = f(b)$ and $f_{ab} = f_{f(b)\cdot b}$. Let $Q = \{f_{ab}$ in $\mathrm{Hom}_R(Rb,Ra)$: $0 \neq b$ in $R\}$.

DEFINITION 1.6: Let $\approx$ be a relation on Q where $f_{ab} \approx f_{cd}$ if and only if there is a nonzero r in R such that $f_{ab}(x) = f_{cd}(x)$ for each x in Rr.

PROPOSITION 1.7: $\approx$ is an equivalence relation.

Proof: As before the reflexive and symmetric laws are immediate and we prove only that $\approx$ is transitive. Let f_{ab}, f_{cd}, and f_{eg} belong to Q with $f_{ab} \approx f_{cd}$ and $f_{cd} \approx f_{eg}$. Thus there exist r' and r'' such that f_{ab} agrees with f_{cd} on Rr' and f_{cd} agrees with f_{eg} on Rr''. Surely they all agree on $Rr' \cap Rr''$ and since $Rr'r'' \subset Rr' \cap Rr''$ and $r'r''$ is not zero, we are through.

As before, we define a set $Q(F)$ to be the set of all of the equivalence classes of Q induced by $\approx$. We denote the equivalence class containing f_{ab} by $[f_{ab}]$.

DEFINITION 1.8: Let $[f_{ab}]$ and $[f_{cd}]$ belong to $Q(F)$. (a) $[f_{ab}] + [f_{cd}] = [f_{ab} \oplus f_{cd}]$ where $f_{ab} \oplus f_{cd} = f_{ab} + f_{cd}$ restricted to Rbd; (b) $[f_{ab}] \cdot [f_{cd}] = [f_{cd} * f_{ab}]$ (note the change in order) where $f_{cd} * f_{ab} = f_{cd} \circ f_{ab}$ (composition) restricted to Rbd.

PROPOSITION 1.9: Addition and multiplication on $Q(F)$ are well-defined.

Proof: Let $[f_{ab}] = [f_{a'b'}]$ and $[f_{cd}] = [f_{c'd'}]$ and denote the principal ideal on which f_{ab} agrees with $f_{a'b'}$ by Rx and denote the principal ideal on which f_{cd} agrees with $f_{c'd'}$ by Ry. Then for all z in Rxy; $f_{ab}(z) = f_{a'b'}(z)$ and

$f_{cd}(z) = f_{c'd'}(z)$ so $f_{ab}(z) + f_{cd}(z) = f_{a'b'}(z) + f_{c'd'}(z)$ and so $(f_{ab}+f_{cd})(z) = (f_{a'b'}+f_{c'd'})(z)$. Thus $f_{ab} + f_{cd} \approx f_{a'b'} + f_{c'd'}$ and $[f_{ab}] + [f_{cd}] = [f_{a'b'}] + [f_{c'd'}]$. Also for z in Rxy, where $z = rxy$, we have $f_{cd}(f_{ab}(rxy)) = f_{cd}(yf_{ab}(rx)) = f_{cd}(yf_{a'b'}(rx)) = f_{cd}(f_{a'b'}(rx)y)$. But since $f_{a'b'}((rx)y)$ is in Ry it is an element on which f_{cd} and $f_{c'd'}$ agree and so $f_{cd}(f_{a'b'}(rx)y) = f_{c'd'}(f_{a'b'}(rx)y) = f_{c'd'}(f_{a'b'}(rxy))$. Hence $f_{cd} \circ f_{ab} \approx f_{c'd'} \circ f_{a'b'}$ and $[f_{ab}] \cdot [f_{cd}] = [f_{a'b'}] \cdot [f_{c'd'}]$.

In the next theorem we use the fact that the isomorphic image of a field is a field, thereby alleviating us of the tedious task of verifying that $(Q(F),+,\cdot)$ is a field. We also obtain the result that $Q(F)$ contains a copy of R.

<u>THEOREM 1.10</u>: $Q(F)$ is isomorphic to Q^*.

<u>Proof</u>: Define $\phi : Q^* \to Q(F)$ by $\phi([a,b]) = [f_{ab}]$. To show that ϕ is well-defined let $[a,b] = [c,d]$ so $ad = bc$ and so $ard = bcr$ for each r in R. Hence f_{ab} and f_{cd} agree on Rbd. To show that ϕ is one-to-one let $[f_{ab}] = [f_{cd}]$ where f_{ab} and f_{cd} agree on Rx. If z is in Rx, then $z = rx = r'b = r''d$. So $f_{ab}(z) = f_{cd}(z)$ implies $ar' = cr''$ and so $ar'bd = cr''bd$. Thus $adz = bcz$ and $(a,b) \sim (cz,dz)$ and $[a,b] = [cz,dz] = [c,d]$. Clearly ϕ is onto and we check that ϕ is a ring homomorphism. If d is in R we claim that $[f_{ab}] \cdot [f_{dd}] = [f_{ab}]$. Choose x in Rbd so that $x = rbd$; then $f_{dd}(f_{ab}(rbd)) = f_{dd}(ard) = ard = f_{ab}(rbd)$. Also if x belongs to Rbd so that $x = rbd$; then $(f_{dd} \circ f_{ab} + f_{bb} \circ f_{cd})(x) = f_{dd}(f_{ab}(rbd)) + f_{bb}(f_{cd}(rbd)) =$

ard + crb = $(f_{ad+bc,bd})(rbd)$. Thus $[f_{ab}] + [f_{cd}] = [f_{ad+bc,bd}]$. Now $\phi([a,b] + [c,d]) = \phi([ad+bc,bd]) = [f_{ad+bc,bd}] = [f_{ab}] + [f_{cd}] = \phi([a,b]) + \phi([c,d])$. Similarly, since $f_{cd}(f_{ab}(x)) = f_{ac,bd}(x)$ for each x in Rbd; $\phi([a,b] \cdot [c,d]) = \phi[ac,bd] = [f_{ac,bd}] = [f_{ab}] \cdot [f_{cd}] = \phi[a,b] \cdot \phi[c,d]$.

CHAPTER II

RINGS OF LEFT QUOTIENTS

In this chapter we shall generalize some of the results we obtained from chapter one to an arbitrary ring R and abandon the restriction of working with integral domains.

Let R and Q be rings and recall that all our rings have unity, which we shall denote by 1. We call a ring homomorphism f from R into Q a <u>unital ring homomorphism</u> if $f(1) = 1$. It is not hard to check that whenever we have a unital ring homomorphism $f : R \to Q$ then we can make Q into a left R-module by defining $a \cdot q = f(a) \cdot q$ for all a in R and q in Q.

<u>DEFINITION 2.1</u>: Let R and Q be rings and ϕ be a unital ring homomorphism from R into Q with ϕ one-to-one. We call (Q,ϕ) a <u>ring of left quotients</u> of R if ${}_RQ$ is an essential extension of $\phi(R)$. We note that ${}_RQ$ is an essential extension of $\phi(R)$ if and only if for each nonzero q in Q there is an a in R such that $\phi(a) \cdot q = a \cdot q \neq 0$ in $\phi(R)$.

<u>DEFINITION 2.2</u>: Let R be a ring. A nonempty set F of left ideals of R is a <u>complete filter</u> provided:

(a) If D is in F and I is a left ideal of R with $D \subset I$, then I is in F.

(b) If D and D' belong to F then $D \cap D'$ is in F.

(c) If D is in F and r is in R, then $(D:r)$ is in F, where $(D:r) = \{a \text{ in } R: ar \text{ is in } D\}$.

(d) If I is a left ideal of R, D belongs to F, and $(I:d)$ is in F for each d in D, then I is in F.

We shall also call a complete filter faithful whenever $\{a \text{ in } R:(0:a) \text{ is in } F\} = 0$. We call this set the F-singular ideal and denote it $Z_F(R)$. We shall show later that $Z_F(R)$ is a two-sided ideal of R for any complete filter F. We also note that $Z_F(R) = 0$ says that whenever a is in R and $D \cdot a = 0$ for some D in F, then $a = 0$. Equivalently, filter elements do not annihilate nonzero elements of the ring.

As a result of the preceding definitions, if one is careful about the left ideals chosen, then one can show that a ring R has a ring of left quotients. In fact we will show that if one chooses a set F of left ideals which form a faithful complete filter, then R has a ring of left quotients with respect to F, which we shall denote by (Q_F, ϕ_F). To that end, let F be a faithful complete filter of left ideals of R and $Q = \{f \text{ in } \mathrm{Hom}_R(D,R): D \text{ is in } F\}$. We define a relation $\sim$ on Q as follows: If f and g belong to Q, then $f \sim g$ if and only if f and g agree on some D in F. As in chapter one it is easy to check that $\sim$ is an equivalence relation, therefore inducing a partition of Q into equivalence classes. If f is in Q, let $[f]$ denote the equivalence class containing f and let $Q_F = \{[f]: f \text{ is in } Q\}$.

For notational convenience, if f is in Q, we shall denote the domain of f by domf. The following lemma is immediate.

<u>LEMMA 2.3</u>: If for each f and g in Q we define the function $f + g : \mathrm{domf} \cap \mathrm{domg} \to R$ by $(f+g)(x) = f(x) + g(x)$ for each x in $\mathrm{domf} \cap \mathrm{domg}$, then $f + g$ belongs to Q.

We also state the following lemma whose proof is given.

<u>LEMMA 2.4</u>: If for each f and g in Q we define the function $f \circ g : \mathrm{domg} \cap g^{-1}(\mathrm{domf}) \to R$ by $f \circ g(x) = f(g(x))$ for each x in $\mathrm{domg} \cap g^{-1}(\mathrm{domf})$, then $f \circ g$ is in Q.

<u>Proof</u>: Let x and y belong to $\mathrm{domg} \cap g^{-1}(\mathrm{domf})$ and let r belong to R. Then $f \circ g(x+y) = f(g(x+y)) = f(g(x) + g(y)) = f(g(x)) + f(g(y)) = f \circ g(x) + f \circ g(y)$. Also $f \circ g(rx) = f(g(rx)) = f(rg(x)) = rf(g(x)) = rf \circ g(x)$. Thus $f \circ g$ is an R-homomorphism. All we have to check is that $\mathrm{domg} \cap g^{-1}(\mathrm{domf})$ is in $\mathcal{F}$. Choose d in domf and consider $(\mathrm{domg} \cap g^{-1}(\mathrm{domf}):d)$. Thus x belongs to $(\mathrm{domg} \cap g^{-1}(\mathrm{domf}):d)$ if and only if xd belongs to $\mathrm{domg} \cap g^{-1}(\mathrm{domf})$. But this last statement is true if and only if $g(xd)$ is in domf and this is true if and only if $xg(d)$ is in domf which is equivalent to x belongs to $(\mathrm{domf}:g(d))$. Thus $(\mathrm{domg} \cap g^{-1}(\mathrm{domf}):d) = (\mathrm{domf}:g(d))$. But $(\mathrm{domf}:g(d))$ belongs to $\mathcal{F}$ and so $(\mathrm{domg} \cap g^{-1}(\mathrm{domf}:d)$ is in $\mathcal{F}$ for each d in domf. Hence $\mathrm{domg} \cap g^{-1}(\mathrm{domf})$ is in $\mathcal{F}$ by part (d) of definition 2.2.

As a result of these two lemmas we know that $[f+g]$ and $[g \circ f]$ are in $Q_{\mathcal{F}}$ whenever $[f]$ and $[g]$ are in $Q_{\mathcal{F}}$. We now define two operations on Q_F. If $[f]$ and $[g]$ belong to Q_F then: (a) $[f] + [g] = [f+g]$, (b) $[f] \cdot [g] = [g \circ f]$ (Note the change of order).

PROPOSITION 2.5: Addition and multiplication on Q_F are well-defined.

Proof: Let $[f]$, $[f']$, $[g]$, and $[g']$ belong to Q_F with $[f] = [f']$ and $[g] = [g']$. Then $f \sim f'$ and $g \sim g'$ so that there are left ideals D and D' in F such that f and f' agree on D and g and g' agree on D'. Hence f agrees with f' on $D \cap D'$ and g agrees with g' on $D \cap D'$, so $f + g$ agrees with $f' + g'$ on $D \cap D'$. Thus $f + g \sim f' + g'$ and so $[f+g] = [f'+g']$. Now let $I = D \cap f^{-1}(D')$ and by lemma 2.4, I is in F. Choose x in I so that $g \circ f(x) = g(f(x)) = g'(f(x)) = g'(f'(x)) = g' \circ f'(x)$. Hence $[f] \cdot [g] = [f'] \cdot [g']$.

PROPOSITION 2.6: $(Q_F, +, \cdot)$ is a ring.

Proof: Let $[f]$, $[g]$, and $[h]$ belong to Q_F.

(a) It is easy to check that $f + g$ and $g + f$ agree on $\mathrm{dom} f \cap \mathrm{dom} g$, so $f + g \sim g + f$; hence $[f+g] = [g+f]$.

(b) It is also easy to check that $f + (g+h)$ agrees with $(f+g) + h$ on $\mathrm{dom} f \cap \mathrm{dom} g \cap \mathrm{dom} h$ and so $[f] + ([g] + [h]) = ([f] + [g]) + [h]$.

(c) Let $[f_0]$ belong to Q_F where $f_0(x) = 0$ for each x in R. Clearly R belongs to F. One checks that $f + f_0$ agrees with f on $\mathrm{dom} f$ and so $[f] + [f_0] = [f]$. It is easy to see that $[f_0]$ is the only equivalence class with this property.

(d) If $[f]$ is in Q_F, we define $-f$ in Q by $-f(x) = -y$ if and only if $f(x) = y$ for each x in domf. Clearly dom$-f$ = domf and so choose x in domf; $(f+(-f))(x) = f(x) + (-f)(x) = 0 = f_0(x)$ so that $f + (-f) \sim f_0$ and so $[f] + [-f] = [f_0]$.

(e) Choose x in domf $\cap f^{-1}$(domg) $\cap g^{-1}$(domh) $\cap$ domg. One checks that $((h \circ g) \circ f)(x) = h \circ (f \circ g)(x)$ and so $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$.

(f) Let $1_{Q_F}(x) = x$ for each x in R. It is easy to see that $f \circ 1_{Q_F} \sim f$ for all f in Q and so $[1_{Q_F}]\,[f] = [f]$.

(g) Finally, let I = domf $\cap f^{-1}$(dom$(g+h)$) and choose x in I. Then $((g+h) \circ f)(x) = (g+h)(f(x)) = g(f(x)) + h(f(x)) = g \circ f(x) + h \circ f(x) = (g \circ f + h \circ f)(x)$ so that $(g+h) \circ f \sim g \circ f + h \circ f$ and $[f] \cdot ([g] + [h]) = [f] \cdot [g] + [f] \cdot [h]$. The other distributive law is similar.

We now establish that $(Q(F), +, \cdot)$ is a ring of left quotients containing a copy of R.

<u>THEOREM 2.7</u>: Let $\phi_F : R \quad Q_F$ be defined by $\phi_F(a) = [f_a]$ where $f_a(x) = xa$ for each x in R. Then ϕ_F is a unital ring homomorphism and $\phi_F(R) \,\underline{\Delta}\, {}_R Q_F$. Further, the kernel of ϕ_F is the F-singular ideal $Z_F(R)$, so that R is isomorphic to $\phi_F(R)$ since F is faithful.

Proof: Clearly $[f_1]$ is the multiplicative identity of Q_F and $\phi_F(1) = [f_1]$. Now $\phi_F(a+b) = [f_{a+b}]$ where $f_{a+b}(x) = xa + xb = f_a(x) + f_b(x)$ for all x in R. So $\phi_F(a+b) = [f_{a+b}] = [f_a] + [f_b] = \phi_F(a) + \phi_F(b)$. Similarly $\phi_F(ab) = \phi_F(a) \cdot \phi_F(b)$ and so ϕ_F is a unital ring homomorphism. We note that a is in Ker ϕ_F if and only if $\phi_F(a) = [f_a]$ where $[f_a] = [f_0]$. Hence there is a D in F such that $Da = f_a(D) = f_0(D) = 0$. Thus Ker $\phi_F = Z_F(R)$. Since F is faithful ϕ_F is one-to-one.

We make Q_F into a left R-module by defining $a[f] = \phi_F(a)[f]$ for all a in R and $[f]$ in Q_F. It is easy to see that the above makes Q into a left R-module and so we procede directly to proving that $\phi_F(R) \underline{\Delta}\ {}_RQ_F$. Choose $[f_0] \neq [f]$ in Q, and $0 \neq r$ in R such that $f(r) \neq 0$ and let $f(r) = a$. Then $[f_r] \cdot [f] = [f \circ f_r]$ where $f_r(x) = xr$. Thus $f \circ f_r(x) = f(f_r(x)) = f(xr) = xf(r) = xa = f_a(x)$ for all x in dom $f_r \cap f_r^{-1}(\text{domf})$. Thus $[f_r] \cdot [f] = [f_a]$. Clearly $[f_a] \neq 0$, for then $f \circ f_r(x) = xa = 0$ for all x in some D in F, so $Da = 0$ and hence $a = 0$, a contradiction.

The preceding discussion has shown that if a ring R has a set of left ideals which form a faithful complete filter, then R has a ring of left quotients. It is natural to ask if a ring has a non-trivial filter, and whether a ring R having one ring of left quotients has more than one, and if so, the relationship between the two. A partial answer to this is avaliable in a theorem first proven by Utumi [1]. We begin with the following definition.

<u>DEFINITION 2.8</u>: A left ideal D of a ring R is <u>dense</u> if and only if for all a and b in R, if $(D:a) \cdot b = 0$, then $b = 0$. It is not hard to check that this definition is equivalent to the following found in [2], which we state as a lemma.

<u>LEMMA 2.9</u>: A left ideal D of a ring R is dense if and only if for each a in R and $0 \neq b$ in R, there is an r in R such that ra is in D and $rb \neq 0$.

<u>PROPOSITION 2.10</u>: The set $\mathcal{D}$ of all dense left ideas of R is a faithful complete filter.

<u>Proof</u>: (a) we must show that if D is in $\mathcal{D}$, and $D \subset I \subset R$ for a left ideal I or R, then I is in $\mathcal{D}$. Clearly if $D \subset I$, then $(D:b) \subset (I:b)$ for each b in R; hence if $(I:b) \cdot a = 0$, then $(D:b) \cdot a = 0$ and so $a = 0$. Thus I belongs to $\mathcal{D}$.

(b) We next show that if D and D' are in $\mathcal{D}$, then $D \cap D'$ is in $\mathcal{D}$. Since D is in $\mathcal{D}$, then for each a and nonzero b in R there is an r in R such that ra is in D and $rb \neq 0$. But since D' is in $\mathcal{D}$ and $rb \neq 0$, there is an r' in R such that $r'(ra)$ is in D' and $r'(rb) \neq 0$. But D is a left ideal and so $r'(ra)$ is in D, hence in $D \cap D'$.

(c) We show if D is in $\mathcal{D}$, and a is in R, then $(D:a)$ is in $\mathcal{D}$. Let D belong to $\mathcal{D}$, and

$0 \neq b$ in R. Then there is an r in R such that ra is in D and $rb \neq 0$. Thus $((D:a):r) \cdot b \neq 0$ since 1 belongs to $((D:a):r)$. Thus $((D:a):r) \cdot b = 0$ implies $b = 0$ and so $(D:a)$ is in $\mathcal{D}$.

(d) We show that $\{a \text{ in } R:(0:a) \text{ is in } \mathcal{D}\} = 0$. If $a = 0$ and $(0:a)$ is in $\mathcal{D}$ we use lemma 2.9 to obtain a contradiction. Recall if $(0:a)$ is in $\mathcal{D}$, then for all a' in R and $0 \neq b'$ in R there is an r' in R such that $r'a'$ is in $(0:a)$ and $r'b' \neq 0$. We simply choose $a' = 1$ and $b' = a$.

(e) Finally we show that if I is a left ideal of R, D is in $\mathcal{D}$, and $(I:d)$ is in $\mathcal{D}$ for each d in D, then I is in $\mathcal{D}$. Since d is in $\mathcal{D}$, then for each a and nonzero b in R there is an r in R such that ra is in D and $rb \neq 0$. Hence $(I:ra)$ is in $\mathcal{D}$ and $(I:ra) \cdot b \neq 0$ and so there is an r' in $(I:ra)$ such that $r'(rb) \neq 0$. Thus I is in $\mathcal{D}$.

As a corollary to part (e) note that if D and D' are in $\mathcal{D}$, then DD' is in $\mathcal{D}$ because if x is in D then xd' is in DD' for each d' in D' and so x is in $(DD':d)$. Thus $D \subset (DD':d')$ and $(DD':d')$ belongs to $\mathcal{D}$ by (a) and DD' is in $\mathcal{D}$ by (e).

We call the ring of left quotients $(Q_{\mathcal{D}}, \phi_{\mathcal{D}})$ given to us by the set of dense left ideals of R the Utumi or maximal ring of left quotients. We shall justify the name "maximal" shortly, but first we establish the following.

PROPOSITION 2.11: If $\mathcal{F}$ is any faithful complete filter and I is in $\mathcal{F}$, then I is a dense left ideal; thus $\mathcal{F} \subset \mathcal{D}$.

Proof: Consider (I:b) in $\mathcal{F}$. If $(I:b) \cdot a = 0$ then $(I:b) \subset (0:a)$ and so $(0:a)$ is in $\mathcal{F}$ and $a = 0$. Thus I belongs to $\mathcal{D}$.

As a result of proposition 2.11 we know that any faithful complete filter is contained in $\mathcal{D}$. In particular, any partial endomorphism f from D to R, where D is in $\mathcal{F}$, is also a partial endomorphism from D to R, where D is in $\mathcal{D}$. If we identify [f] in $Q_{\mathcal{F}}$ with the equivalence class in $Q_{\mathcal{D}}$ to which f belongs we will have proven the following theorem.

THEOREM 2.12: Let $\mathcal{F}$ be any faithful complete filter. The ring of left quotients of R induced by $\mathcal{F}$, $(Q_{\mathcal{F}}, \phi_{\mathcal{F}})$, can be embedded in the Utumi ring, $(Q_{\mathcal{D}}, \phi_{\mathcal{D}})$.

Proof: Define $h : Q_{\mathcal{F}} \to Q_{\mathcal{D}}$ by $h([f]) = [f_{\mathcal{D}}]$ where $[f_{\mathcal{D}}]$ is the equivalence class to which f belongs. Clearly h is well-defined since if $[f] = [g]$ in $Q_{\mathcal{F}}$ then f and g agree on some I in $\mathcal{F}$ and so f and g agree with some I in $\mathcal{D}$. Hence f and g belong to the same equivalence class in $Q_{\mathcal{D}}$; alternatively $[f_{\mathcal{D}}] = [g_{\mathcal{D}}]$. Similarly one checks that h is a ring homomorphism.

CHAPTER III

CLASSICAL RINGS OF LEFT QUOTIENTS

Having shown that $(Q_{\mathcal{D}}, \phi_{\mathcal{D}})$ is a maximal ring of left quotients, we now turn our attention to a special type of ring of left quotients, namely, the classical ring of left quotients of a ring R. As we shall see, not every ring has a classical ring of left quotients and we begin our discussion of this topic by defining what we mean by a classical ring of left quotients.

<u>DEFINITION 3.1</u>: Let R be a ring and Q_c an overring of R with $h : R \to Q_c$ a one-to-one ring homomorphism. We call (Q_c, h) a <u>classical ring of left quotients of R</u> provided: (a) $h(d)^{-1}$ is in Q_c for all d in $U(R)$; (b) For each nonzero q in Q_c there is a d in $U(R)$ such that $h(d) \cdot q$ is in $h(R)$.

We note that if Q_c is a classical ring of left quotients of R, then $Q_c = \{h(d)^{-1} \cdot h(a) : a \text{ is in } R, d \text{ in } U(R)\}$.

The preceding remarks do not assert anything about the existence of a classical ring of left quotients of a ring. In fact, all we know at this point is what the elements of a classical ring of left quotients of a ring R look like provided the ring R has one. We procede to show the conditions under which a ring will have a classical ring of left quotients. We begin with the following definition.

DEFINITION 3.2: A ring R is called a left Öre ring if for each r in R and d in $U(R)$ there is an r' in R and d' in $U(R)$ such that $d'r = r'd$.

THEOREM 3.3: If a ring R has a classical ring of left quotients, then R is a left Öre ring.

Proof: Let (Q_c, h) be a classical ring of left quotients of R. For each a in R and d in $U(R)$, $h(a)$ and $h(d)$ belong to Q_c and so $h(d)^{-1}$ is in Q_c. Hence $h(a) \cdot h(d)^{-1}$ is in Q_c and so there is an a' in R and d' in $U(R)$ such that $h(a) \cdot h(d)^{-1} = h(d')^{-1} \cdot h(a')$. Hence $h(d') \cdot h(a) = h(a') \cdot h(d)$ and $h(d'a) = h(a'd)$. Thus $d'a = a'd$ since h is one-to-one and R is a left Öre ring.

The next proposition shows us how we can obtain a classical ring of left quotients from a left Öre ring.

THEOREM 3.4: Let R be a left Öre ring and $F = \{I \subset R : I$ is a left ideal of R and $I \cap U(R) \neq \emptyset\}$, then F is a faithful complete filter and (Q_F, ϕ_F) is a classical ring of left quotients.

Proof: First we show that F is a faithful complete filter.

(a) We first show if I is in F, and $I \subset D$ for some left ideal D of R, then D is in F. Since $I \cap U(R) \neq \emptyset$ there is a nonzero divisor d in I, hence in D, hence $D \cap U(R) \neq \emptyset$ and so D is in F.

(b) We show if D and D' are in F, then $D \cap D'$ is in F. Since D belongs to F, there is a nonzero divisor

d in $D\ U(R)$ with $d \neq 0$. Similarly, since D' is in $\mathcal{F}$ there is a d' in $D' \cap U(R)$. By the Öre condition there exist a' in R and d'' in $U(R)$ such that $d''d' = a'd$. But D is a left ideal and so $d''d'$ is in D, hence in $D \cap D'$. Thus $d''d'$ is in $(D\ D') \cap U(R)$ and $D \cap D'$ is in $\mathcal{F}$.

(c) We show if D is in $\mathcal{F}$, then $(D{:}a)$ is in $\mathcal{F}$ for each a in R. D in $\mathcal{F}$ imples the existence of a nonzero divisor d in $D \cap U(R)$. By the Öre condition, for each a in R and d in $U(R)$ there is an a' in R and a d' in $U(R)$ such that $d'a = a'd$. Again, D is a left ideal and so $d'a$ is in D, hence d' is in $(D{:}a)$. Thus d' is in $(D{:}a) \cap U(R)$ and $(D{:}a)$ is in $\mathcal{F}$.

(d) Next we show if D is in $\mathcal{F}$, $(I{:}a)$ is in $\mathcal{F}$, for each a in D, and I is a left ideal of R, then I is in $\mathcal{F}$. Since D belongs to $\mathcal{F}$ there is a d in $D \cap U(R)$ hence $(I{:}d)$ belongs to $\mathcal{F}$. Thus there is a d' in $(I{:}d) \cap U(R)$. In particular d' is in $(I{:}d)$ and so $d'd$ is in $I \cap U(R)$.

(e) Finally we show $Z_{\mathcal{F}}(R) = 0$. If $(0{:}a)$ is in $\mathcal{F}$ then there is a d in $(0{:}a) \cap U(R)$, hence $d \cdot a = 0$ and $a = 0$.

Surely $(Q_{\mathcal{F}}, \phi_{\mathcal{F}})$ is a ring of left quotients and all that we need to show is that it is a classical ring of left quotients. For each d in $U(R)$, $\phi_{\mathcal{F}}(d) = [f_d]$ in Q. Consider $f' : Rd \to R$

where $f'(rd) = r$. We claim that $\phi_F(d)^{-1} = [f']$. $[f'][f_d] = [f_d \circ f']$ where $f_d \circ f'(x) = f_d(f'(x))$ for all x in $\operatorname{dom} f' \cap f'^{-1}(\operatorname{dom} f_d)$, and so $f_d(f'(x)) = f_d(f'(rd)) = f_d(r) = rd = x$ where $x = rd$ in Rd. So $f_d \circ f' \sim f_1$ and $[f'] \cdot [f_d] = [f_1]$. Similarly one shows that $[f_d][f'] = [f_1]$ and so $[f'] = [f_d]^{-1} = \phi_F(d)^{-1}$. Finally if $0 \neq [f]$ in Q_F, let d belong to $\operatorname{dom} f \cap U(R)$ and let $f(d) = a$. $[f_d][f] = [f \circ f_d]$ where $f \circ f_d(x) = f(f_d(x))$ for all x in $\operatorname{dom} f_d \cap f_d^{-1}(\operatorname{dom} f)$ and so $f(f_d(x)) = f(xd) = xf(d) = xa = f_a(x)$. Thus $f \circ f_d \sim f_a$ and $[f_d][f] = [f_a]$.

The results of the preceding two theorems establishes a necessary and sufficient condition for a ring R to have a classical ring of left quotients. We conclude the paper with a uniqueness theorem regarding the classical ring of left quotients of a left Öre ring.

<u>THEOREM 3.5</u>: Any two classical rings of left quotients of a left Öre ring R are isomorphic.

<u>Proof</u>: Let (Q_c, ϕ_c) and (Q_F, ϕ_F) be classical rings of left quotients of the ring R. We shall show that there is an isomorphism between Q_c and Q_F. Let q, q_1, and q_2 belong to Q_c where $q = \phi_c(x)^{-1} \cdot \phi_c(y)$, $q_1 = \phi_c(d)^{-1} \cdot \phi_c(c)$, and $q_2 = \phi_c(b)^{-1} \cdot \phi_c(a)$ with x, d, and b in $U(R)$. For notational convenience we shall identify R with its isomorphic image $\phi_c(R)$ and write "$d^{-1} \cdot c$ in Q_c" instead of the more cumbersome $\phi_c(d)^{-1} \cdot \phi_c(c)$. Similarly we write "$d^{-1} \cdot c$ in Q_F" instead of

$\phi_F(d)^{-1} \cdot \phi_F(c)$. This notation causes no ambiguity since both ϕ_c and ϕ_F are one-to-one functions. We define $f : Q_c \to Q_F$ as follows: f maps $d^{-1} \cdot c$ in Q_c to $d^{-1} \cdot c$ in Q_F. Since $c = a$ in Q_c if and only if $c = a$ in R and $c = a$ in R if and only if $c = a$ in Q_F, then $c = a$ in Q_c if and only if $c = a$ in Q_F. A similar argument shows that f is well-defined and one-to-one. Clearly f is onto and we now show that f is a ring homomorphism. If $b^{-1}a$ and $d^{-1}c$ belong to Q_c, we have $b^{-1}a + d^{-1}c = (d'b)^{-1}(d'a + b'c)$ where we use the Öre condition to write $d'b = b'd$ with d' in $U(R)$. Also, $(d^{-1}c) \cdot (b^{-1}a) = (b'd)^{-1}(c'a)$ where $b'c = c'b$ and b' is in $U(R)$. Thus, $f(b^{-1}a + d^{-1}c) = f((d'b)^{-1}(d'a+b'c)) = (d'b)^{-1}(d'a+b'c)$ in Q_F. But $(d'b)^{-1} \cdot (d'a+b'c)$ in Q_F is, in fact, equal to $b^{-1}a + d^{-1}c$ where each of these is in Q_F. A similar argument shows that f preserves products and we are through.

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